Propagator in the instanton background and instanton-induced processes in scalar model

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Abstract

The cross section of the multiparticle scattering processes in the non-perturbative sector of the scalar $(-\lambda)\phi^4$ model is studied within the semiclassical approximation. For this purpose the exact formula for the residue of the propagator in the instanton background is derived. The exponent of the cross section is calculated at small energies both analytically and numerically in the leading and next-to-leading orders in energy. The results are in agreement with the numerical result of Ref. [1].

1 Introduction

In the present paper we study non-perturbative contributions to scattering processes in the simple scalar model. The motivation is that analogous processes play an important role in more realistic theories. One class of such processes includes winding number transitions between topologically inequivalent vacua in non-abelian gauge theories and sigma models. In a number of papers (for a review see, e.g., Refs.[2]) instanton transitions induced by particle collisions in the Electroweak Theory were studied. In QCD, instanton-like processes in the deep inelastic scattering and their possible experimental detection are under intensive investigation now [3].

The decay of metastable (false) vacuum [4] is another well-known example where tunneling plays an important role. It occurs in scalar theories. The solution which interpolates between initial and final state is the bounce configuration [5]. The bounce solution has one turning point; the field configuration at the turning point represents the final state after the false vacuum decay has taken place. At low energies the amplitude of the false vacuum decay is proportional to $\exp(-S_B/2)$, where $S_B/2$ is the Euclidean action of the bounce configuration calculated up to the turning point (S_B is the action of the whole bounce solution).

Finally, there are so-called shadow processes first considered in Refs. [6, 7]. These are the processes we concentrate on in this paper. They arise in models with false vacuum decay and correspond to transitions between initial and final states, both lying in the false vacuum, through the intermediate state containing the bubble of the true vacuum. At low energies, such transitions are dominated by the same bounce solution, the difference with the false vacuum decay being that the final state is not the one corresponding to the turning point, but to asymptotically large Euclidean times. Thus, amplitudes of such processes are proportional to $\exp(-S_B)$. Clearly these processes are unphysical since their probabilities are much smaller than the probability of the false vacuum decay. Nevertheless, formally these processes are fully analogous to the instanton-like transitions in gauge theories and, because of relative simplicity of scalar models, can serve as a good laboratory for testing methods of calculation of probabilities of instanton-like transitions.

The main method for studying instanton-like transitions at non-zero energies is the generalization of the semiclassical approximation. It can be shown that instanton contribution into the total cross section of the inclusive process $N \rightarrow any$ with the initial energy E is given by the semiclassical expression [8, 9]

$$\sigma_N(E) = \sum_M \sigma_{N \rightarrow M}(E) \sim e^{\frac{1}{\lambda} F(\epsilon, \nu) + \mathcal{O}(\lambda^0)}, \tag{1}$$

where λ is the coupling constant, $\epsilon = E/E_{sph}$, $\nu = N/N_{sph}$, and E_{sph}, N_{sph} are the sphaleron energy and the sphaleron number of particles, respectively[1]. In the limit of very low energy of initial particles the exponent in Eq. (1) is equal to

$$\frac{1}{\lambda} F(0,0) = -2 S_{inst} \tag{2}$$

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[1] The number of particles in the sphaleron is defined as the number of particles to which the sphaleron decays if slightly perturbed [10]. In the case of false vacuum decay or shadow processes the analog of the sphaleron is a critical bubble [4].

and the cross section is suppressed by the inverse power of the small coupling constant. However, at parametrically high energy $E \sim E_{sph} \sim 1/\lambda$, the cross section depends exponentially on E. This leads to the reduction of the suppression factor at energies of the order of the sphaleron energy, so that instanton induced processes may become significant. The main application of this approach is based on the conjecture of Refs. [8, 9] that in the limit $\nu \to 0$ the function $F(\epsilon, \nu)$ reproduces the exponent in the instanton contribution into the total cross section of the *two-particle* scattering and thus allows to estimate the latter.

The induced false vacuum decay in the $(-\lambda)\phi^4$ model was studied in Ref. [1]. There the function $F(\epsilon, \nu)$ was computed numerically in the range of parameters $0.4 \leq \epsilon \leq 3.5$ and $0.25 \leq \nu \leq 1.0$. The calculation was based on a numerical realization of the formalism of Refs. [8] - [11] involving solution of a certain classical boundary value problem. In the present paper we study the case of shadow processes. We limit ourselves to perturbation theory around the instanton. Although such perturbation theory works only at $E \ll E_{sph}$ and does not provide an explicit procedure for calculating the function $F(\epsilon, \nu)$ in a closed form, it still remains one of the main theoretical tools for studying instanton-like transitions.

Similar to the conventional perturbation theory, the key role in the perturbative expansion around the instanton is played by the propagator in the instanton background. The scattering amplitudes, as well as the contributions to the function $F(\epsilon, \nu)$, are expressed through the on-mass-shell residues of the propagator and vertices in the external instanton field. While in most models possessing instantons (in particular, in the Electroweak Theory) the propagator in the instanton background is not known explicitly, this is not the case in the model we consider here. Making use of this advantage, we first compute exactly the residue of the propagator in the instanton background found in Ref. [12]. We then calculate the leading and next-to-leading corrections to the zero energy value (2) of the function $F(\epsilon, \nu)$. This approximation is valid for low energies, and besides being useful on its own, enables one to obtain the behavior of the exponential factor in the range of parameters which was not covered by the numerical computations in Ref. [1]. In the low energy limit analytical expressions for $F(\epsilon, \nu)$ can be obtained. The latter allows us to check (in this approximation) the validity of the conjecture of Refs. [8, 9] about the relation between two-particle and multiparticle cross sections. This conjecture essentially relies on the regularity of the function $F(\epsilon, \nu)$ in the limit $\nu \to 0$, and we show that indeed all singular terms, appearing at the intermediate stage of the calculation, cancel out in the final expression.

This paper is organized as follows. In Sect. 2 we describe the model and calculate

the residue of the propagator in the instanton background. In Sect. 3 we calculate the function $F(\epsilon, \nu)$ in the leading and next-to-leading approximations. The results are compared to numerical results of Ref. [1] in the region where the latter can be translated to the case of shadow processes. Analysis of the expression for the suppression factor in the limit of small ϵ and ν will also be given. Sect. 4 contains discussion and concluding remarks.

2 Propagator in the instanton background and its residue

Consider the model of one real scalar field with the action

$$S = \int d^4x \left(\frac{1}{2}\partial_\mu\phi\partial^\mu\phi - \frac{1}{2}m^2\phi^2 + \frac{\lambda}{4!}\phi^4 \right) \tag{3}$$

in the four-dimensional Minkowski space-time, where $\lambda > 0$. For $m \neq 0$ the potential has the meta-stable minimum at $\phi = 0$ and is unbounded from below when $|\phi| \to \infty$.

In the case $m = 0$ the state $\phi = 0$ is stable classically but is unstable with respect to quantum fluctuations. Its decay is described by the instanton solutions [13, 14],

$$\phi_{inst}(x) = \frac{4\sqrt{3}}{\sqrt{\lambda}} \frac{\rho}{(x - x_0)^2 + \rho^2} \tag{4}$$

parameterized by the size ρ and position x_0. Due to conformal invariance of the theory (3) in the massless case, the size ρ can be arbitrary and the action of the instanton configuration

$$S_{inst} = \frac{16\pi^2}{\lambda}$$

does not depend on it. For the perturbative calculations of the function $F(\epsilon, \nu)$ the on-mass-shell residue of the instanton solution will be needed. By definition it equals

$$R_{inst}(\mathbf{p}) = \frac{1}{(2\pi^2)^{3/2}\sqrt{2w_{\mathbf{p}}}} \lim_{p^2 \to -m^2} (p^2 + m^2)\tilde{\phi}_{inst}(p),$$

where $\tilde{\phi}_{inst}(p)$ is the Fourier transform of the instanton,

$$\tilde{\phi}_{inst}(p) = \int d^4x\, e^{ipx} \phi_{inst}(x).$$

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From explicit expression (4) one obtains

$$\tilde{\phi}_{inst}(p) = \frac{16\sqrt{3}\pi^2}{\sqrt{\lambda}} \frac{\rho^2}{|p|} K_1(\rho|p|), \tag{5}$$

where $K_1(z)$ is the modified Bessel function, and

$$R_{inst}(\mathbf{p}) = \frac{1}{\sqrt{\lambda}} \frac{16\sqrt{3}\pi^2 \rho}{(2\pi^2)^{3/2}\sqrt{2w_{\mathbf{p}}}}. \tag{6}$$

For $m \neq 0$ regular Euclidean solutions with finite action do not exist. The decay of the meta-stable state $\phi = 0$ is dominated by approximate solutions known as constrained instantons [15]. They minimize the Euclidean action of the theory under the constraint that the size of the configuration is ρ. The conformal invariance is broken in the massive case and the action depends on ρ and on the constraint. For $\rho^2 m^2 \ll 1$ the constraint instanton configuration coincides with the massless instanton (4) at $|x| \ll \rho$ and decreases as

$$\phi_{inst}(x) \sim \frac{2\sqrt{6}\pi}{\sqrt{\lambda}} \frac{\rho\sqrt{m}}{|x|^{3/2}} e^{-m|x|}$$

for $|x| \geq m^{-1}$. In this case the constrainet-independent part of the instanton action has the form [1]

$$S_{inst} = \frac{16\pi^2}{\lambda} - \frac{24\pi^2}{\lambda}(\rho m)^2 \left[\ln\frac{(\rho m)^2}{4} + 2C_E + 1\right] + \mathcal{O}\left((\rho m)^4\right), \tag{7}$$

where $C_E = 0.577\ldots$ is the Euler constant (notice the difference in the normalization of the coupling constant in Eq. (3) and in Ref. [1]). The constraint dependence appears only in the terms of the order $\mathcal{O}\left((\rho m)^4\right)$ and higher. Similarly, other quantities of interest, in particular, the residues of the instanton field and propagator in the instanton background, are series in the parameter $m^2\rho^2$. To our approximation only the leading term is important, and thus we can calculate residues in the massless theory.

In the massive case the height of the barrier separating the vacuum $\phi = 0$ and the instability region is finite and is determined by a static sphaleron configuration found numerically in [1]. From this solution one gets

$$E_{sph} = \kappa\frac{m}{\lambda}, \quad N_{sph} = \frac{\delta}{\lambda},$$

where the numerical factors κ and δ are equal to $\kappa = 113.4$, $\delta = 63$.

Now let us turn to the discussion of the propagator in the instanton background and to calculation of its residue. The expression for the instanton propagator in the massless case was obtained in Ref. [12] and used there for the computation of the first correction to the asymptotic formula for the large-order coefficients of the perturbation theory expansion [14], [16].

The equation for the propagator $G(x, y)$ in the background of the instanton (4) reads

$$\left\{-\frac{\partial}{\partial x^\mu}\frac{\partial}{\partial x^\mu} - \frac{\lambda}{2}\phi_{inst}^2(x)\right\}G(x, y) = \delta^4(x - y) - \sum_{A=1}^{5}\frac{\rho^2\psi_A(x)\psi_A(y)}{(x^2 + \rho^2)^2}, \tag{8}$$

where $\psi_A(x)$ are zero modes of quadratic action describing fluctuations around the instanton.

In calculation of the propagator in [12] the $O(5)$-symmetry of the theory (3) with $m = 0$ is essential. After making the stereographic projection of the four-dimensional Euclidean space E^4 onto the sphere S^4, embedded into the five-dimensional Euclidean space, Eq. (8) simplifies drastically and becomes the equation for the free propagator on S^4. Then, representing the propagator in terms of the spherical harmonics and using the summation formulas, one derives the explicit expression for $G(x, y)$. Perhaps it is not surprising that tools used for the computation of large-order coefficients of perturbation theory find also their application in the computation of the non-perturbative cross-section at high energies. Deep connections between these two problems have been observed in Refs. [17].

The propagator in the instanton background in terms of the coordinates in E^4 is equal to

$$\begin{aligned} G(x, y) &= \frac{1}{2\pi^2}\frac{\rho^2}{(\rho^2 + x^2)(\rho^2 + y^2)}\{\frac{3t^2(x, y) - 1}{2}\frac{1}{1 - t(x, y)} \\ &- 3t(x, y)\ln\frac{1 - t(x, y)}{2} - \frac{41}{10}t(x, y) - \frac{3}{2}\}, \end{aligned} \tag{9}$$

where as before ρ is the size of the instanton. The function $t(x_1, x_2)$ is defined through the geodesic distance $d(\xi_1, \xi_2)$ between the points ξ_1 and ξ_2 on the sphere S^4 which correspond to the points x_1 and x_2 in the four-dimensional Euclidean space:

$$t(x_1, x_2) = \cos(d(\xi_1, \xi_2)).$$

In terms of the coordinates in E^4 the function $t(x_1, x_2)$ equals to

$$t(x_1, x_2) = 1 - \frac{2\rho^2(x_1 - x_2)^2}{(\rho^2 + x_1^2)(\rho^2 + x_1^2)}. \tag{10}$$

The limit when points x_1 and x_2 coincide corresponds to $t(x_1, x_2) \to 1$. The propagator (9) is singular in this limit, as it should be. For $x \to y$ its leading singularity is given by

$$G(x, y) \sim \frac{1}{2\pi^2} \frac{\rho^2}{(\rho^2 + x^2)(\rho^2 + y^2)} \frac{1}{1 - t(x, y)} + \text{weaker singularities.}$$

Using Eq. (10) the leading singularity can be transformed into the expression which coincides with the free propagator of a scalar field:

$$G_0(x, y) = \frac{1}{4\pi^2} \frac{1}{(x - y)^2}.$$

This result is quite clear. Indeed, it can be shown that the diagram, corresponding to the propagator in the instanton background, can be represented as an infinite sum of diagrams D_n. Each D_n consists of a line, representing the free propagator, with n insertions of two instanton fields each, so that D_n has two external lines of the quantum free field and $2n$ external lines of the classical instanton field. Analytically the relation between the full instanton propagator and the sum of the free propagators with the insertions is written as follows:

$$
\begin{aligned}
G(x_1, x_2) &= G_0(x_1, x_2) + \frac{\lambda}{2} \int dy G_0(x_1, y)\phi_{inst}^2(y) G_0(y, x_2) + \ldots \\
&= \frac{1}{4\pi^2} \frac{1}{(x_1 - x_2)^2} - \frac{3}{2\pi^2} \frac{\rho^2}{(\rho^2 + x_1^2)(\rho^2 + x_2^2) - \rho^2(x_1 - x_2)^2} \\
&\quad \times \ln \frac{\rho^2(x_1 - x_2)^2}{(\rho^2 + x_1^2)(\rho^2 + x_2^2)} + \ldots
\end{aligned}
$$

The first term, which is just the free propagator, corresponds to the diagram D_0 and is the most singular one. The subleading singularity, given by the second term, corresponds to D_1 and is of the logarithmic type, in agreement with the complete expression (9).

As it was already said in the Introduction, for the calculation of the exponential factor in Eq. (1) in the next-to-leading approximation one needs the double on-mass-shell residue $R(p_1, p_2)$ of the propagator in the instanton background. This function

is defined as

$$R(p_1, p_2) = \lim_{p_{1,2}^2 \to -m_{1,2}^2} (p_1^2 + m_1^2)(p_2^2 + m_2^2)\tilde{G}(p_1, p_2), \qquad (11)$$

where $\tilde{G}(p_1, p_2)$ is the standard Fourier transform,

$$\tilde{G}(p_1, p_2) = \int d^4x\, d^4y\, G(x, y) \exp(ip_1 x + ip_2 y).$$

In the massless approximation the residue $R(p_1, p_2)$ is actually a function of $\rho^2 s$ only, where the variable s is defined by $s \equiv s(p_1, p_2) = (p_1 + p_2)^2 = 2(p_1 p_2)$.

Calculation of $R(p_1, p_2)$ is rather tedious although straightforward procedure, and we do not give the details here. However, we would like to discuss some general features of the computation before presenting the answer.

The terms in the expression (9) give contributions to the residue which can be divided in four classes.

1) The first term in the curly brackets in eq. (9) gives rise to the free propagator term as it was already explained. Its contribution to the Fourier transform $\tilde{G}(p_1, p_2)$ is proportional to

$$\frac{2(2\pi)^4}{p_1^2}\delta(p_1 + p_2).$$

This describes free motion of the particle not interacting with the instanton and is irrelevant for our problem.

2) There are factorizable terms of the form $f_1(x)f_2(y)$, where $f_i(x)$'s are proportional to expressions like

$$\frac{x^n}{(\rho^2 + x^2)^m} \quad \text{or} \quad \frac{x^n \ln(\rho^2 + x^2)}{(\rho^2 + x^2)^m} \qquad (12)$$

with some integer n and m. Their contributions to the momentum-space propagator are of the form $\tilde{f}_1(p_1^2)\tilde{f}_2(p_2^2)$. These obviously give s - independent contributions to the residue (11).

3) The next group of terms are of the form $(xy)f_1(x)f_2(y)$ with f_i given by (12). Calculating the momentum-space propagator we get

$$\int e^{ip_1 x + ip_2 y}(x, y)f_1(x)f_2(y) = -\frac{\partial}{\partial p_1^\mu}\frac{\partial}{\partial p_2^\mu} \int e^{ip_1 x + ip_2 y} f_1(x)f_2(y)$$

$$= -\frac{\partial}{\partial p_1^\mu}\frac{\partial}{\partial p_2^\mu}\tilde{f}_1(p_1^2)\tilde{f}_2(p_2^2) = -4(p_1, p_2)\tilde{f}_1'(p_1^2)\tilde{f}_2'(p_2^2).$$

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This gives a contribution to the residue proportional to $s(p_1, p_2)$.

4) The last group consists of terms of the form $(xy) \ln(x - y)^2 f_1(x) f_2(y)$ and $\ln(x - y)^2 f_1(x) f_2(y)$. Carrying out the computations one can show that they lead to terms proportional to $s \ln s$ and $\ln s$ respectively in the expression for the function (11).

Finally, the expression for the residue of the momentum-space propagator with both momenta on the mass shell is equal to

$$R(s) = 16\pi^2 \left[\frac{3}{4} s \ln \frac{s}{\sqrt{2}} + \frac{3}{2} s \left(C_E - \frac{1}{15} \right) - \frac{3}{2} \ln \frac{s}{\sqrt{2}} + 3 \left(C_E - \frac{43}{30} \right) \right]. \qquad (13)$$

This result agrees with the asymptotic formula for large s derived in Ref. [18]:

$$R(s) = \lim_{p_{1,2}^2 \to 0} p_1^2 p_2^2 \frac{\tilde{\phi}_{inst}(p_1) \tilde{\phi}_{inst}(p_2)}{(\partial_\mu \phi_{inst}, r^{-2} \partial^\mu \phi_{inst})} (p_1, p_2) \ln(p_1, p_2). \qquad (14)$$

Indeed, using the expression (5) for the Fourier transform of the instanton solution and working out the formula (14) one can see that it coincides with the leading asymptotic of the exact expression (13) for large s:

$$R(s) \sim 12\pi^2 s \ln s.$$

3   Multiparticle cross section

The formula (1) for the multiparticle cross-section of shadow processes comes from the following expression derived in Ref. [9],

$$\sigma_N(E) \sim \int d^4x d\rho d^4\xi d\theta \exp\left[-2S_{inst}(\rho) + \frac{1}{\lambda} W^{(1)}(x_0, \rho, \xi, \theta) \right.$$
$$+ \left. \frac{1}{\lambda} W^{(2)}(x_0, \rho, \xi, \theta) + \dots \right], \qquad (15)$$

where we integrate over the position x and the size ρ of the instantons, as well as over auxiliary variables ξ_μ and θ. We also indicated explicitly the dependence of the action on the size of the instanton (see Eq. (7)). The terms $W^{(i)}$ account for fluctuations in the instanton background: $W^{(1)}$ corresponds to leading diagrams without propagator insertions, $W^{(2)}$ corresponds to diagrams with one internal propagator in the instanton background, etc. Diagrams with loops do not appear in the $(1/\lambda)$ order of the semi-classical approximation, they contribute to $\mathcal{O}(\lambda^0)$ terms in (1). The integrals in (15)

are evaluated by the saddle point method. Simple dimensional analysis shows that the result (the function F in (1)), actually depends on $\epsilon = E/E_{sph}$ and $\nu = N/N_{sph}$ only.

In the present paper we limit ourselves to the calculation of the first two contributions $W^{(1)}$ and $W^{(2)}$. General expressions for these functions, derived in [9], are quite cumbersome and are given in the Appendix. It can be seen that up to the next-to-leading order the saddle point values of x, ρ, ξ and θ are determined by the leading-order equations. These equations are obtained by differentiation of the expression $S_{inst} + W^{(1)}/\lambda$, where S_{inst} and $W^{(1)}$ are given by the expressions (7) and (A2), with respect to x_0, ρ, ξ and θ. The physically relevant saddle point has $x_i = 0$, $\xi_i = 0$ ($i = 1, 2, 3$), while x_0, ξ_0 and θ are purely imaginary. It is convenient to introduce the following notations,

$$x_0 = i\tau, \quad \xi_0 = i\chi, \quad \theta = -i\ln\gamma. \tag{16}$$

For general values of ϵ and ν, the system of saddle point equations is transcendental and cannot be solved analytically even in the leading order. It simplifies considerably in the limit of small ν. To the leading order in ν the saddle point solution can be written in the form

$$\tilde{\rho}^2 = -\frac{\kappa}{192\pi^2 m^2}\frac{\epsilon}{\Phi'(\tilde{\tau})}, \tag{17}$$

$$\tilde{\gamma} = -\frac{1}{4}\left(\frac{\delta\nu}{\kappa\epsilon}\right)^3\Phi'(\tilde{\tau}), \tag{18}$$

$$\tilde{\chi} = \tilde{\tau} + \frac{2}{m}\frac{\delta\nu}{\kappa\epsilon}, \tag{19}$$

where the function $\tilde{\tau} = \tilde{\tau}(\epsilon, \nu)$ is determined by the equation

$$\ln\left(-\frac{\kappa\epsilon Ce}{192\pi^2\Phi'(\tilde{\tau})}\right) + 4\left[\Phi(\tilde{\tau}) - \frac{\delta\nu}{\kappa\epsilon}\Phi'(\tilde{\tau}) + \mathcal{O}(\nu^2)\right] = 0.$$

Here

$$\Phi(\tau) \equiv \frac{m}{\tau}K_1(m\tau),$$

the prime denotes the derivative with respect to $m\tau$ and $C = -\ln 4 + 2C_E + 1$.

It is straightforward although lengthy calculation to substitute these expressions into the exponent in Eq. (15) and check that the limit $\nu \to 0$ is smooth and singular

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terms appearing at the intermediate stages of the calculation cancel in the final answer. We will show this in the low energy limit. At $\epsilon \to 0$, the saddle point equations simplify further. The solution for $\tilde{\tau}(\epsilon, \nu)$ is

$$\tilde{\tau} = \tilde{\tau}_0(\epsilon) + \delta\nu \frac{8}{\kappa m \epsilon \left(8 - 3(m\tilde{\tau}_0(\epsilon))^2\right)},$$

where $\tilde{\tau}_0(\epsilon)$ is determined by the equation

$$\ln\left(\kappa\epsilon \frac{Ce}{394\pi^2}(m\tilde{\tau}_0)^3\right) = -\frac{4}{(m\tilde{\tau}_0)^2}.$$

The last equation can be solved iteratively, and one gets

$$m\tilde{\tau}_0(\epsilon) = \frac{2}{\sqrt{\ln\frac{1}{\epsilon}}} + \frac{\ln\ln\frac{1}{\epsilon}}{\left(\ln\frac{1}{\epsilon}\right)^{3/2}} + \dots$$

Using relations (17) - (19) one finds in the leading order

$$(m\tilde{\rho})^2 = \frac{1}{48\pi^2}\frac{\kappa\epsilon}{\left(\ln\frac{1}{\epsilon}\right)^{3/2}},$$

$$\tilde{\gamma} = \left(\frac{\delta\nu}{\kappa\epsilon}\right)^3\left(\ln\frac{1}{\epsilon}\right)^{3/2},$$

$$\tilde{\chi} - \tilde{\tau} = \frac{2}{m}\frac{\delta\nu}{\kappa\epsilon}.$$

To the leading order in energy and zeroth order in ν the function $F(\epsilon, \nu)$ can be written as

$$F(\epsilon, \nu) = -32\pi^2 + F^{(1)}(\epsilon, \nu) + F^{(2)}(\epsilon, \nu) + \dots,$$

where

$$F^{(1)}(\epsilon, \nu) = 2\frac{\kappa\epsilon}{\sqrt{\ln\frac{1}{\epsilon}}}\left[1 + \mathcal{O}\left(\frac{\ln\ln\frac{1}{\epsilon}}{\ln\frac{1}{\epsilon}}\right)\right] + \mathcal{O}(\nu) \qquad (20)$$

and

$$F^{(2)}(\epsilon, \nu) = F^{(2)}_{(f-f)}(\epsilon, \nu) + F^{(2)}_{(i-f)}(\epsilon, \nu) + F^{(2)}_{(i-i)}(\epsilon, \nu)$$

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is the sum of contributions coming from final-final, initial-final and initial-initial particle interactions,

$$F^{(2)}_{(f-f)}(\epsilon,\nu) = -\frac{1}{128\pi^2}(\kappa\epsilon m\tilde{\tau}_0)^2 \left[\frac{58}{15} + \ln\left(\frac{\kappa\epsilon m\tilde{\tau}_0}{384\pi^2}\right)\right], \tag{21}$$

$$F^{(2)}_{(i-f)}(\epsilon,\nu) = \frac{6}{(192\pi^2)^2}(\kappa\epsilon m\tilde{\tau}_0)^3 \left[\frac{71}{30} + \ln\left(\frac{(\kappa\epsilon m\tilde{\tau}_0)^2}{768\pi^2}\right) + \ln\frac{1}{\delta\nu}\right], \tag{22}$$

$$F^{(2)}_{(i-i)}(\epsilon,\nu) = -\frac{3}{(192\pi^2)^2}(\kappa\epsilon m\tilde{\tau}_0)^3 \left[\frac{71}{30} + \ln\left(\frac{(\kappa\epsilon m\tilde{\tau}_0)^3}{1536\pi^2}\right) + 2\ln\frac{1}{\delta\nu}\right]. \tag{23}$$

We see that $F^{(2)}_{(i-i)}$ and $F^{(2)}_{(i-i)}$ contain terms $\ln(1/\nu)$ which are singular in the limit $\nu \to 0$. However, when (21) - (23) are summed together the singular terms cancel each other. Finally, we get

$$F^{(2)}(\epsilon,\nu) = \frac{(\kappa\epsilon)^2}{32\pi^2}\left(1 + \frac{1}{2}\frac{\ln\ln\frac{1}{\epsilon}}{\ln\frac{1}{\epsilon}}\right) + \ldots \tag{24}$$

From Eqs. (20), (A3) we see that our approximation is valid as long as

$$\frac{\ln\ln\frac{1}{\epsilon}}{\ln\frac{1}{\epsilon}} \ll 1. \tag{25}$$

At higher energies the condition (25) breaks down, and the calculation has to be done numerically. We have performed this calculation for the values of ν not subject to the condition $\nu \ll 1$. For the periodic instanton case (the values of ϵ and ν such that $\tilde{\gamma} = 1$ and $\tilde{\chi} = 2\tilde{\tau}$) the results can be compared to those obtained in Ref.[1] and show good agreement. Moreover, the next-to-leading order correction $F^{(2)}(\epsilon,\nu)$ improves systematically the agreement as compared to the previous order expression $(-32\pi^2) + F^{(1)}(\epsilon,\nu)$.

4   Discussion and conclusions

In the present paper we have analyzed the multiparticle cross section of the shadow processes induced by instanton transitions in the simple scalar model (3). We have obtained the exact analytical expression for the on-shell residue of the propagator of quantum fluctuations in the instanton background. This allowed us to calculate the

suppression factor in the next-to-leading order. The range of validity of this approximation can be estimated by comparing the results with numerical computations of the complete function $F(\epsilon, \nu)$ in Ref. [1] in the range where the latter can be translated to the case of shadow processes (i.e., for periodic instantons). The comparison shows that the perturbative results do not differ significantly from the exact ones for $\epsilon \leq 0.25$ and $\nu \leq 0.2$. This range can be taken as the region of validity of our perturbative calculation.

For very small energies, namely when the condition (25) holds, we have obtained the analytical expressions for the suppression factor and values of the saddle point parameters ρ, χ, τ and θ. This enables us to check the validity of the approximation used. For instance, the constraint-dependent contributions to the action (and thus to the function $F(\epsilon, \nu)$) are of the order $(\rho m)^4 \sim \epsilon^2/\ln^3(1/\epsilon)$ and are subleading as compared to the terms retained in the function $F^{(2)}(\epsilon, \nu)$, Eq. (24). Also, this calculation allowed us to check explicitly the cancelation of the terms singular in the limit $\nu \to 0$ in the propagator contribution $F^{(2)}(\epsilon, \nu)$.

Acknowledgments

Appendix A

Here we present the expressions for the functions $W^{(1)}$ and $W^{(2)}$ of Eq. (15) in terms of the variables τ, χ and γ (see Eq. (16)). The function $W^{(1)}$ reads

$$\frac{1}{\lambda} W^{(1)}(\tau, \rho, \chi, \gamma) = E\chi - N \ln \gamma + R_b^* \frac{T}{1 - \gamma X} R_b \qquad (A1)$$

$$+ \ \gamma R_a^* \frac{X T^{-1}}{1 - \gamma X} R_a + \gamma R_a \frac{X}{1 - \gamma X} R_b + \gamma R_a^* \frac{X}{1 - \gamma X} R_b^*,$$

where R_a and R_b are the residues of the instanton field on the mass shell corresponding to initial and final particles, respectively (in the notations of Ref. [9]). Here

$$X(\mathbf{p}, \mathbf{k}) = \delta(\mathbf{p} - \mathbf{k}) e^{-\omega_p \chi}, \quad T(\mathbf{p}, \mathbf{k}) = \delta(\mathbf{p} - \mathbf{k}) e^{-\omega_p \tau},$$

and integration

$$\int \frac{d\mathbf{p}\, d\mathbf{k}}{\sqrt{\omega_{\mathbf{p}}\omega_{\mathbf{k}}}}$$

is assumed where appropriate. For our model $R_a(\mathbf{p}) = R_b(\mathbf{p})$ and equals the expression (6) with $\omega_{\mathbf{p}} = \sqrt{\mathbf{p}^2 + m^2}$, so that (A1) can be written as

$$\frac{1}{\lambda}W^{(1)}(\tau, \rho, \chi, \gamma) = E\chi - N\ln\gamma$$

$$+\quad \frac{192\pi^2\rho^2}{\lambda}\left[J(\gamma, \tau, \chi) + \gamma J(\gamma, \chi - \tau, \chi) + 2\gamma J(\gamma, \chi, \chi)\right], \tag{A2}$$

where

$$J(\gamma, \tau, \chi) = \frac{1}{4\pi}\int \frac{d\mathbf{k}}{\omega_{\mathbf{k}}}\frac{e^{-\omega_{\mathbf{k}}\tau}}{1 - \gamma e^{-\omega_{\mathbf{k}}\chi}}.$$

The next-to-leading order function $W^{(2)}$ can be written as the sum of contributions representing interactions of final-final, initial-final and initial-initial particles, respectively,

$$W^{(2)} = W^{(2)}_{(f-f)} + W^{(2)}_{(i-f)} + W^{(2)}_{(i-i)}, \tag{A3}$$

where

$$\frac{1}{\lambda}W^{(2)}_{(f-f)} = \frac{1}{2}\left[R_b \frac{T}{1-\gamma X}D^\dagger_{bb}\frac{T}{1-\gamma X}R_b \right.$$

$$\left. +R_b^* \frac{T}{1-\gamma X}D_{bb}\frac{T}{1-\gamma X}R_b^* + \dots\right],$$

$$\frac{1}{\lambda}W^{(2)}_{(i-f)} = \frac{\gamma}{2}\left[R_b \frac{T}{1-\gamma X}D^\dagger_{ab}\frac{XT^{-1}}{1-\gamma X}R_a \right.$$

$$\left. +R_a^* \frac{XT^{-1}}{1-\gamma X}D_{ab}\frac{T}{1-\gamma X}R_b^* + \dots\right],$$

$$\frac{1}{\lambda}W^{(2)}_{(i-i)} = \frac{\gamma^2}{2}\left[R_a \frac{XT^{-1}}{1-\gamma X}D^\dagger_{aa}\frac{XT^{-1}}{1-\gamma X}R_a \right.$$

$$\left. +R_a^* \frac{XT^{-1}}{1-\gamma X}D_{aa}\frac{XT^{-1}}{1-\gamma X}R_a^* + \dots\right].$$

Only terms relevant in the small ν limit are shown explicitly. D_{aa}, D_{ab} and D_{bb} are related to the double on-mass-shell residue of the propagator, Eq.(13), as follows,

$$D_{\#}(\mathbf{p}, \mathbf{k}) = \frac{\rho^2}{(2\pi)^3 2\sqrt{\omega_{\mathbf{p}}\omega_{\mathbf{k}}}}R(\rho^2 s_{\#}(\mathbf{p}, \mathbf{k})),$$

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where $\# = aa,\ ab,\ bb$ and $s_\#$ is the s-variable for corresponding particles on the mass shell,

$$
\begin{aligned}
s_{aa}(\mathbf{p},\mathbf{k}) &= s_{bb}(\mathbf{p},\mathbf{k}) = (p+k)^2 = 2m^2 - 2[\omega_{\mathbf{p}}\omega_{\mathbf{k}} - \mathbf{pk}], \\
s_{ab}(\mathbf{p},\mathbf{k}) &= (p-k)^2 = 2m^2 + 2[\omega_{\mathbf{p}}\omega_{\mathbf{k}} - \mathbf{pk}].
\end{aligned}
$$

One has

$$
\begin{aligned}
W^{(2)}_{(f-f)} &= 384\pi^2(\rho m)^4\Big[J_{bb}(\gamma,\tau,\tau,\chi) + 2\gamma J_{bb}(\gamma,\tau,\chi,\chi) + \\
&\qquad \gamma^2 J_{bb}(\gamma,\chi,\chi,\chi)\Big], \\
W^{(2)}_{(i-f)} &= 768\pi^2(\rho m)^4\gamma\Big[J_{ab}(\gamma,\tau,\chi,\chi) + J_{ab}(\gamma,\tau,\chi-\tau,\chi) \\
&\quad + \gamma J_{ab}(\gamma,\chi,\chi,\chi) + \gamma J_{ab}(\gamma,\chi-\tau,\chi,\chi)\Big], \\
W^{(2)}_{(i-i)} &= 384\pi^2(\rho m)^4\gamma^2\Big[J_{aa}(\gamma,\chi-\tau,\chi-\tau,\chi) + J_{aa}(\gamma,\chi-\tau,\chi,\chi) \\
&\quad + J_{bb}(\gamma,\chi,\chi,\chi)\Big],
\end{aligned}
$$

where

$$
J_\#(\gamma,\tau_1,\tau_2,\chi) = \frac{1}{8\pi^2}\int \frac{d\mathbf{k}}{\omega_{\mathbf{k}}}\frac{d\mathbf{q}}{\omega_{\mathbf{q}}}\frac{e^{-\omega_{\mathbf{k}}\tau_1}}{1-\gamma e^{-\omega_{\mathbf{k}}\chi}}\frac{R\left(\rho^2 s_\#(\mathbf{k},\mathbf{q})\right)}{16\pi^2}\frac{e^{-\omega_{\mathbf{q}}\tau_2}}{1-\gamma e^{-\omega_{\mathbf{q}}\chi}}.
$$

For $m\tau_{1,2} \ll 1$ and $\gamma = 0$ one finds

$$
\begin{aligned}
J_{aa}(0,\tau_1,\tau_2,\chi) &= J_{bb}(0,\tau_1,\tau_2,\chi) \\
&= 384\pi^2\rho^4\left[-\frac{3}{\tau_1^2\tau_2^2}\left(\frac{58}{15}+\ln\frac{\rho^2}{\tau_1\tau_2}\right) - \frac{12\rho^2}{\tau_1^3\tau_2^3}\left(\frac{71}{30}+\ln\frac{\rho^2}{\tau_1\tau_2}\right)\right], \\
J_{ab}(0,\tau_1,\tau_2,\chi) &= 384\pi^2\rho^4\left[-\frac{3}{\tau_1^2\tau_2^2}\left(\frac{58}{15}+\ln\frac{\rho^2}{\tau_1\tau_2}\right) + \frac{12\rho^2}{\tau_1^3\tau_2^3}\left(\frac{71}{30}+\ln\frac{\rho^2}{\tau_1\tau_2}\right)\right].
\end{aligned}
$$

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